

Towards solving generic cosmological singularity problem

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Abstract. The big bounce transition of the quantum FRW model in the setting of loop quantum cosmology is presented. We determine the physical self-adjoint Hamiltonian generating the dynamics. It is used to define, via the Stone theorem, an evolution operator. We examine properties of expectation values of physical observables in the process of the quantum big bounce transition. The dispersion of observables are studied in the context of the Heisenberg uncertainty principle. We suggest that the real nature of the bounce may become known only after we quantize the Belinskii-Khalatnikov-Lifshitz scenario, which concerns the generic cosmological singularity.

Keywords: Big bounce cosmology, quantum evolution, BKL scenario

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INTRODUCTION

Observational cosmology suggests that the Universe has been expanding for almost 14 billion years and emerged from a state with extremely high energy densities of matter fields. Theoretical cosmology shows that almost all known general relativity models of the Universe (Lemaître, Kasner, AdS, Friedmann, Bianchi, ..., BKL) predict an existence of cosmological singularities with blowing up gravitational and matter fields invariants. The existence of the cosmological singularities in solutions to GR means that this classical theory is *incomplete*. One expects that quantization may heal the singularities.

In what follows we use the canonical quantization of GR based on loop geometry. We apply the so-called *nonstandard* LQC, i.e. the reduced phase space quantization approach ('first solve constraints then quantize') in LQC, which has been developed recently (see, e.g. [1, 2, 3, 4, 5, 6] and refereces therein).

REDUCED PHASE SPACE QUANTIZATION

Modified Hamiltonian

The gravitational part of the Hamiltonian (in the setting of LQC) reads

$$H_g := \int_{\Sigma} d^3x (N^i C_i + N^a C_a + NC) \approx 0, \quad (1)$$

where Σ , space-like part of spacetime $R \times \Sigma$; (N^i, N^a, N) , Lagrange multipliers; (C_i, C_a, C) are Gauss, diffeomorphism and scalar constraints; $(a, b = 1, 2, 3)$, spatial indices; $(i, j, k = 1, 2, 3)$, internal $SU(2)$ indices.

For flat FRW universe with massless scalar field we have

$$H_g = -\gamma^{-2} \int_{\mathcal{V}} d^3x N e^{-1} \varepsilon_{ijk} E^{aj} E^{bk} F_{ab}^i, \quad (2)$$

where γ , the Barbero-Immirzi parameter; $\mathcal{V} \subset \Sigma$, elementary cell; N , lapse function; ε_{ijk} , alternating tensor; E_i^a , density weighted triad; $F_{ab}^k = \partial_a A_b^k - \partial_b A_a^k + \varepsilon_{ij}^k A_a^i A_b^j$, curvature of $SU(2)$ connection A_a^i ; $e := \sqrt{|\det E|}$.

Making use of Thiemann's identity leads finally to

$$H_g = \lim_{\lambda \rightarrow 0} H_g^{(\lambda)}, \quad H_g^{(\lambda)} = -\frac{\text{sgn}(p)}{2\pi G \gamma^3 \lambda^3} \sum_{ijk} N \varepsilon^{ijk} \text{Tr} \left(h_i^{(\lambda)} h_j^{(\lambda)} (h_i^{(\lambda)})^{-1} (h_j^{(\lambda)})^{-1} h_k^{(\lambda)} \{ (h_k^{(\lambda)})^{-1}, V \} \right), \quad (3)$$

where $h_k^{(\lambda)}$ is holonomy of connection around a loop with size λ , and $V = |p|^{\frac{3}{2}} = a^3 V_0$ is the volume of the elementary cell \mathcal{V} . Variables c and p determine connections A_a^k and triads E_k^a as follows: $A_a^k = {}^o\omega_a^k c V_0^{-1/3}$ and $E_k^a = {}^o e_k^a \sqrt{q_o} p V_0^{-2/3}$, where $c = \gamma \dot{a} V_0^{1/3}$, $|p| = a^2 V_0^{2/3}$, and $\{c, p\} = 8\pi G \gamma / 3$; scale factor is a , and \dot{a}/a , Hubble parameter.

The total Hamiltonian for FRW universe with a massless scalar field ϕ is given by

$$H = H_g + H_\phi, \quad H_\phi := p_\phi^2 |p|^{-\frac{3}{2}} / 2, \quad (4)$$

where ϕ and p_ϕ are elementary variables satisfying $\{\phi, p_\phi\} = 1$. The relation $H \approx 0$ enables defining physical phase space. Finally, the total Hamiltonian corresponding to (4) reads

$$H^{(\lambda)} / N = -\frac{3}{8\pi G \gamma^2} \frac{\sin^2(\lambda \beta)}{\lambda^2} v + \frac{p_\phi^2}{2v}, \quad \beta := \frac{c}{|p|^{1/2}}, \quad v := |p|^{3/2}, \quad (5)$$

where $\beta \sim \dot{a}/a$ and $v \sim a^3$, for $\lambda = 0$. Eq (5) presents a *modified* classical Hamiltonian. It includes no quantum physics.

Dirac observables

A function \mathcal{O} defined on phase space is called the Dirac observable if it weakly Poisson commutes with the constraint: $\{\mathcal{O}, H^{(\lambda)}\} \approx 0$. One can show that in the *physical* phase space we have only two elementary observables satisfying the algebra: $\{\mathcal{O}_2, \mathcal{O}_1\} = 1$.

Compound observables are functions of elementary ones. They are supposed to be *measurable* observables. In what follows we consider the volume in space:

$$v = |w|, \quad w := \kappa \gamma \lambda \mathcal{O}_1 \cosh 3\kappa(\phi - \mathcal{O}_2). \quad (6)$$

Quantization problem of v reduces to the quantization of w :

$$\hat{w}f(x) := \kappa \gamma \lambda \frac{1}{2} (\hat{\mathcal{O}}_1 \cosh 3\kappa(\phi - \hat{\mathcal{O}}_2) + \cosh 3\kappa(\phi - \hat{\mathcal{O}}_2) \hat{\mathcal{O}}_1) f(x), \quad (7)$$

where $f \in L^2(R)$. For \mathcal{O}_1 and \mathcal{O}_2 we use the Schrödinger representation:

$$\mathcal{O}_1 \longrightarrow \hat{\mathcal{O}}_1 f(x) := -i\hbar \partial_x f(x), \quad \mathcal{O}_2 \longrightarrow \hat{\mathcal{O}}_2 f(x) := \hat{x}f(x) := xf(x). \quad (8)$$

Thus, an explicit form of \hat{w} reads

$$\hat{w} = i \frac{\kappa \gamma \lambda \hbar}{2} \left(2 \cosh 3\kappa(\phi - x) \frac{d}{dx} - 3\kappa \sinh 3\kappa(\phi - x) \right). \quad (9)$$

Solution to the eigenvalue problem:

$$\hat{w} f_a(x) = a f_a(x), \quad a \in R, \quad (10)$$

is found to be [2]:

$$f_a(x) := \frac{\sqrt{\frac{3\kappa}{\pi}} \exp\left(i \frac{2a}{3\kappa^2 \gamma \lambda \hbar} \arctan e^{3\kappa(\phi-x)}\right)}{\cosh^{\frac{1}{2}} 3\kappa(\phi-x)}, \quad a = b + 6\kappa^2 \gamma \lambda \hbar m = b + 8\pi G \gamma \lambda \hbar m, \quad (11)$$

and where $b \in R$, $m \in \mathbb{Z}$. Completion of the span of

$$\mathcal{F}_b := \{ f_a \mid a = b + 8\pi G \gamma \lambda \hbar m \} \subset L^2(R), \quad (12)$$

in the norm of $L^2(R)$ leads to $L^2(R)$, $\forall b \in R$. The operator \hat{w} is essentially *self-adjoint* on each span of \mathcal{F}_b .

Due to the relation (6) and the spectral theorem on self-adjoint operators we get the solution of the eigenvalue of the volume operator:

$$v = |w| \longrightarrow \hat{v}f_a := |a|f_a. \quad (13)$$

The spectrum is *bounded* from below and *discrete*. There exists the minimum gap $\Delta := 8\pi G\gamma\hbar\lambda$ in the spectrum, which defines a *quantum* of the volume. In the limit $\lambda \rightarrow 0$, corresponding to the classical FRW model, there is no quantum of the volume.

Quantization of another compound observable, the energy density ρ , is presented in our paper [2]. The spectrum of the corresponding operator is *continuous* and *bounded* from above. As $\lambda \rightarrow 0$ the energy density blows up, $\rho \rightarrow \infty$, which corresponds to the classical case.

EVOLUTION OF QUANTUM SYSTEM

To define an *evolution* of the universe in a quantum phase, we identify first the so-called *true* Hamiltonian, H . It is obtain by inserting the constraint into Hamilton's equations and finding a new (true) Hamiltonian that leads to these equations of motion. One finds

$$H_\lambda = \frac{2}{\lambda\sqrt{G}}P\sin(\lambda Q), \quad (14)$$

where $P := v/(4\pi l_P^2\gamma)$ and $Q := \beta$, and where $\{Q, P\} = 1$. In the Schrödinger representations for these variables we have:

$$\hat{Q}\phi(Q) := Q\phi(Q), \quad \hat{P}\phi(Q) := -i\frac{d}{dQ}\phi(Q). \quad (15)$$

The quantum Hamiltonian corresponding to (14) reads

$$\hat{H}_\lambda \psi = -\frac{i}{\lambda\sqrt{G}} \left(2\sin(\lambda Q) \frac{d}{dQ} + \lambda \cos(\lambda Q) \right) \psi, \quad (16)$$

where $\psi \in D \subset \mathcal{H} := L^2([0, \pi/\lambda], dQ)$, and where D is some dense subspace of \mathcal{H} .

The eigenvalue problem, $\hat{H}_\lambda \Psi_E = E\Psi_E$, has the solution [5]:

$$\Psi_E(x) = \sqrt{\frac{\lambda\sqrt{G}}{4\pi} \cosh\left(\frac{2}{\sqrt{G}}x\right)} e^{iEx}, \quad E \in R, \quad (17)$$

where $x := \frac{\sqrt{G}}{2} \ln \left| \tan\left(\frac{\lambda Q}{2}\right) \right|$.

We specify the domain of \hat{H}_λ as follows

$$D(\hat{H}_\lambda) := \text{span}\{\varphi_k, k \in Z\}, \quad \varphi_k(Q) := \int_{-\infty}^{\infty} c_k(E) \Psi_E(Q) dE, \quad c_k \in C_0^\infty(R). \quad (18)$$

One can prove that \hat{H}_λ is an essentially *self-adjoint* operator on $D(\hat{H}_\lambda)$.

The classical Hamiltonian H is positive-definite because $\lambda \in [0, \pi]$ and $P \in [0, \infty)$. The corresponding self-adjoint operator \hat{H} has however eigenvalues $E \in R$. We therefore introduce a *physical* Hamiltonian \hat{H} , which has only nonnegative eigenvalues. It is defined as follows

$$\hat{H}\Psi_E := |E|\Psi_E, \quad E \in R. \quad (19)$$

Using Stone's theorem we define an unitary operator of the evolution:

$$\hat{U}(s) = e^{-is\hat{H}}, \quad (20)$$

where $s \in R$ is a time parameter. The state at any moment of time reads

$$|\Psi(s)\rangle = \hat{U}(s)|\Psi(0)\rangle = e^{-is\hat{H}}|\Psi(0)\rangle. \quad (21)$$

Let us consider a superposition of the Hamiltonian eigenstates $|\Psi(0)\rangle = \int_{-\infty}^{+\infty} dE c(E) |\Psi_E\rangle$ at $s = 0$. Then, evolution of this state is given by

$$|\Psi(s)\rangle = \int_{-\infty}^{+\infty} dE c(E) e^{-isE} |\Psi_E\rangle. \quad (22)$$

We consider the Gaussian packet with a simple profile defined to be $c(E) := \sqrt[4]{2\alpha/\pi} \exp\{-\alpha(E - E_0)^2\}$, that is centered at E_0 with the dispersion parameterized by α . The normalized packet corresponding to (22) reads

$$\Psi(x, s) = \sqrt{\frac{\lambda \cosh\left(\frac{2}{\sqrt{G}}x\right)}{\sqrt{8\pi\tilde{\alpha}}}} e^{-\frac{(x-s)^2}{4\alpha}} e^{iE_0(x-s)}, \quad (23)$$

where $\tilde{\alpha} := \alpha/G$.

One can determine an evolution of the *dispersions* $\Delta\hat{Q}$ and $\Delta\hat{P}$, and the product $\Delta\hat{Q} \Delta\hat{P}$. To see the corresponding plots, we recommend our recent paper [6]. It turns out that the Heisenberg uncertainty relation, $\Delta\hat{Q} \Delta\hat{P} \geq 1/2$, is perfectly *satisfied* during the entire evolution.

CONCLUSIONS

The cosmic singularity problem of FRW model can be *resolved* by using the loop geometry: big bang turns into big bounce. The *discreteness* of the spectra of the volume operator may favor a *foamy* structure of space at short distances that may be detected in astro-cosmo observations. The *evolution* of a quantum phase can be described in terms of a self-adjoint physical Hamiltonian. The Heisenberg uncertainty relation is perfectly satisfied during the *entire* evolution of the universe.

The great challenge is *quantization* of the Belinskii-Khalatnikov-Lifshitz (BKL) scenario [7, 8, 9]. One shows that the FRW metric is dynamically *unstable* in the evolution towards the singularity (breaking of isotropy and homogeneity). The BKL scenario concerns an evolution of spacetime near the cosmological *singularity*. It is a *general* solution to GR in the sense that it corresponds to non-zero measure subset of all initial conditions. It is also *stable* against perturbation of initial conditions. Non-singular quantum BKL theory might be used as a *realistic* model of the very early Universe. Quantization of simple cosmological models like FRW carried out during the last decade may be treated as warming up before meeting this challenge.

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